Reasoning with Graded Notions

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Outline

Logic throughout the history

Wikipedia: Logic is the branch of philosophy concerned with the use and study of valid reasoning

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Logic started as a part of philosophy

All men are mortal and Socrates in a man, therefore Socrates is mortal

Logic throughout the history

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Since 19th century a part of logic has evolved into mathematical logic

$$\mathrm{PA} \nvdash \neg\exists w \mathrm{Proof}(w, `\overline{0} = \overline{1}\text{'})$$

Logic throughout the history

Wikipedia: Logic is the branch of philosophy concerned with the use and study of valid reasoning

Nowadays logic is applied mainly in computer science

$$\vdash [\alpha](x = 4) \to [\alpha; (x := 2x)](x = 8)$$

Logic throughout the history

Wikipedia: Logic is the branch of philosophy concerned with the use and study of valid reasoning

However its role in the study of human reasoning has been diminished

What? How? Why now?

Stenning (psychologist) and van Lambalgen (logician) advocate the indispensability of logic for the study of human reasoning

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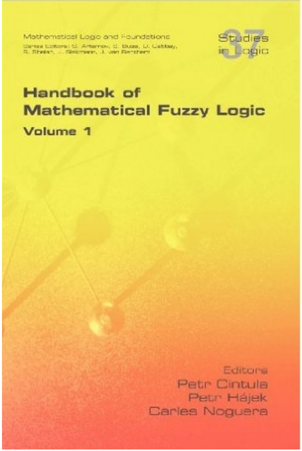
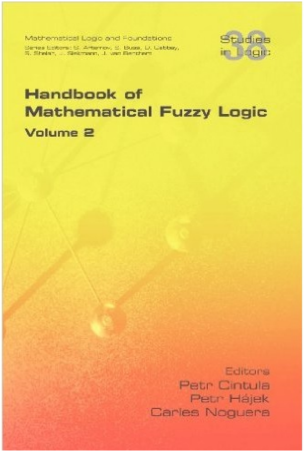
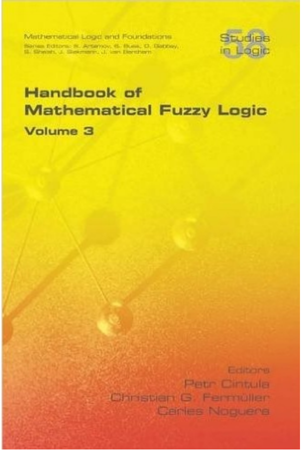
Mathematical logic of graded notions is well developed

As witnessed by 1300 pages of . . .







I have a vision

The three layers of my vision

I To study natural reasoning scenarios involving graded notions
and their natural transformations into formalized scenarios
preserving the graduality

II To utilize logical methods to analyze and perform reasoning
in formalized scenarios

III To advance the logic of graded notions to meet the demands
of the previous goals

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Advancement of mathematical logic

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More powerful formal methods (mainly for CS)

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Advancement of mathematical logic

The three layers of my vision and their three outcomes (one more ambitious than the other :))

I To study natural reasoning scenarios involving graded notions and their natural transformations into formalized scenarios preserving the graduality

Better understanding of (human) reasoning

II To utilize logical methods to analyze and perform reasoning in formalized scenarios

More powerful formal methods (mainly for CS)

III To advance the logic of graded notions to meet the demands of the previous goals

Advancement of mathematical logic

And now a much more humble goals of this talk . . .

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And accompany it with an example of a formalized reasoning scenario

Outline

What are the ‘notions’? (DC 1)

A language is a quadruple $\mathfrak{L} = \langle \mathbf{C}, \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$
connectives, predicate and function symbols with their arities

And we build sets of $\mathfrak{L}$-terms Term and $\mathfrak{L}$-formulas Form as usual,
i.e., as the least sets such that:

- object variables ObjVar are terms
- if $t_1, \ldots, t_n \in$ Term, $F \in \mathbf{F}$, $\mathbf{ar}(F) = n$, then $F(t_1, \ldots, t_n) \in$ Term
- if $t_1, \ldots, t_n \in$ Term, $P \in \mathbf{P}$, $\mathbf{ar}(P) = n$, then $P(t_1, \ldots, t_n) \in$ Form
- if $\varphi_1, \ldots, \varphi_n \in$ Form, $c \in \mathbf{C}$, $\mathbf{ar}(c) = n$, then $c(\varphi_1, \ldots, \varphi_n) \in$ Form
- if $\varphi \in$ Form and $x \in$ ObjVar, then $(\forall x)\varphi \in$ Form and $(\exists x)\varphi \in$ Form

More on languages

There is a well-known difference between the role of connectives and other syntactical objects.

Let us fix, for this talk, a set of connectives $\mathbf{C}$ and their arities

Thus we can speak about predicate languages $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

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Thus we can speak about predicate languages $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

We also consider a special language: the propositional one

$$\mathcal{L} = \langle \mathbf{C}, \{p_i \mid i \in \mathrm{N}\}, \emptyset, \mathbf{ar} \rangle, \quad \text{where } \mathbf{ar}(p_i) = 0$$

Note that $\mathcal{L}$ can be seen as an algebraic type

i.e., a classical predicate language $\langle \emptyset, \mathbf{C}, \mathbf{ar} \rangle$

Where are the grades coming from? (DC 2,3)

I want my semantics to assign some ‘grades’ from a set G to formulas:

$$\|\cdot\|\colon \text{Form} \to G$$

Let us also fix the ‘interpretation’ of connectives, i.e., operations

$$c^{\boldsymbol{G}}\colon G^n \to G \qquad \text{for each } n\text{-ary } c \in \mathbf{C}$$

Then we simply set (DC 2)

$$\|c(\varphi_1, \dots, \varphi_n)\| = c^{\boldsymbol{G}}(\|\varphi_1\|, \dots, \|\varphi_n\|)$$

The classical structure $\boldsymbol{G} = \langle G, \langle c^{\boldsymbol{G}} \rangle_{c \in \mathbf{C}} \rangle$ is then an algebra of type $\mathcal{L}$

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Finally, let us assume that G is partially ordered (DC3)

some grades are 'better' than others

Generalized semantics (DC 3′, 4)

Consider a 'normal' predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ DC1

Consider an algebra $\boldsymbol{G}$ of type $\mathcal{L}$ with a partial order $\leq$ DC2, DC3

$\boldsymbol{G}$-structure $\mathfrak{M}$ for $\mathcal{P}$ is tuple $\mathfrak{M} = \langle M, \langle f_{\mathfrak{M}} \rangle_{f \in \mathbf{F}}, \langle P_{\mathfrak{M}} \rangle_{P \in \mathbf{P}} \rangle$ where

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$\mathfrak{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{ObjVar} \to M$; extended to all terms/fle:

$$\|f(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} = f_{\mathfrak{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) \quad \text{for } f \in \mathbf{F}$$

$$\|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} = P_{\mathfrak{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) \quad \text{for } P \in \mathbf{P}$$

$$\|c(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} = c^{\boldsymbol{G}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) \quad \text{for } c \in \mathbf{C}$$

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$\|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}}$	$=$	$P_{\mathfrak{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}})$	for $P \in \mathbf{P}$
$\|c(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}}$	$=$	$c^{\boldsymbol{G}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}})$	for $c \in \mathbf{C}$
$\|(\forall x)\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}}$	$=$	$\inf_{\leq}\{\|\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}$	DC4
$\|(\exists x)\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}}$	$=$	$\sup_{\leq}\{\|\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}$	DC4

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Consider a 'normal' predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ DC1

Consider an algebra $\boldsymbol{G}$ of type $\mathcal{L}$ with a lattice reduct DC2, DC3′

$\boldsymbol{G}$-structure $\mathfrak{M}$ for $\mathcal{P}$ is tuple $\mathfrak{M} = \langle M, \langle f_{\mathfrak{M}} \rangle_{f \in \mathbf{F}}, \langle P_{\mathfrak{M}} \rangle_{P \in \mathbf{P}} \rangle$ where

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$$\|(\forall x)\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}} = \inf_{\leq}\{\|\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\} \quad \text{DC4}$$

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Example

Take the standard MV-algebra $[0,1]_{Ł} = \langle [0,1], \&, \to, \wedge, \vee, 0, 1\rangle$ where

$$x \,\&\, y = \max\{x+y-1, 0\} \qquad x \to y = \min\{1-x+y, 1\}$$

$$x \wedge y = \min\{x, y\} \qquad x \vee y = \max\{x, y\}$$

Consider a $[0,1]_{Ł}$-structure with domain $M = \{1,\dots,6\}$ and binary predicate P: 'x likes y':

$P_{\mathfrak{M}}$	1	2	3	4	5	6
1	1.0	1.0	0.5	0.4	0.3	0.0
2	0.8	1.0	0.4	0.4	0.3	0.0
3	0.7	0.9	1.0	0.8	0.7	0.4
4	0.9	1.0	0.7	1.0	0.9	0.6
5	0.6	0.8	0.8	0.7	1.0	0.7
6	0.3	0.5	0.6	0.4	0.7	1.0

$\mathrm{Narciss}(R) \equiv_{\mathrm{df}} (\forall x)Rxx$ $\qquad \|\mathrm{Narciss}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{Sym}(R) \equiv_{\mathrm{df}} (\forall x,y)(Rxy \to Ryx)$ $\qquad \|\mathrm{Sym}(P)\|^{\mathfrak{M}} = 0.4$

$\mathrm{Trans}(R) \equiv_{\mathrm{df}} (\forall x,y,z)(Rxy \,\&\, Ryz \to Rxz)$ $\qquad \|\mathrm{Trans}(P)\|^{\mathfrak{M}} = 1$

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5	0.6	0.8	0.8	0.7	1.0	0.7
6	0.3	0.5	0.6	0.4	0.7	1.0

$\mathrm{ELS}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(Rxy)$ $\qquad \|\mathrm{ELS}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{EILS}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(Ryx)$ $\qquad \|\mathrm{EILS}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{ELSE}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(x \neq y \wedge Rxy)$ $\qquad \|\mathrm{ELSE}(P)\|^{\mathfrak{M}} = 0.7$

Consequence (DC5)

Definition ((Sentential) consequence relation)

Let $\boldsymbol{G}$ be an $\mathcal{L}$-algebra with lattice reduct.
Let $T \cup \{\varphi\}$ be a set of $\mathcal{P}$-formulas.
Then φ is a semantical consequence of T w.r.t. $\boldsymbol{G}$, $T \models_{\boldsymbol{G}} \varphi$, if

each $\boldsymbol{G}$-model of T is $\boldsymbol{G}$-model of φ

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Assume, from now on, that $\mathcal{L}$ contains a nullary connective $\overline{1}$ DC5

$\mathfrak{M}$ is a $\boldsymbol{G}$-model of $\mathfrak{M} \models T$ if for each $\mathfrak{M}$-evaluation v:

- $\|\chi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each formula χ and
- $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{G}}$ for each $\varphi \in T$ DC5

$\overline{1}^{\boldsymbol{G}}$ is the least 'good' grade

Consequence (DC5)

Definition ((Sentential) consequence relation)

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras with lattice reduct.
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Then φ is a semantical consequence of T w.r.t. $\mathbb{K}$, $T \models_{\mathbb{K}} \varphi$, if

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Examples

$\mathbb{K}$	logic
$\{\mathbf{2}\}$	classical FOL
(complete) Boolean algebras	classical FOL
(complete) Heyting algebras	intuitionistic FOL
(complete) SI Heyting algebras	intuitionistic FOL + CD
(complete) Heyting chains	int. FOL + $(\varphi \to \psi) \vee (\psi \to \varphi)$ + CD
Gödel algebras	int. FOL + $(\varphi \to \psi) \vee (\psi \to \varphi)$
(complete) $\mathrm{FL_{ew}}$-algebras	affine FOL (w/o expon.)
MV-algebras	Łukasiewicz FOL

- SI Heyting algebras = Heyting algebras with a coatom
- CD: $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$ (x not free in χ)
- Gödel algebras = variety generated by Heyting chains
- MV-algebras = variety generated by $[0, 1]_{Ł}$

How the propositional logics look like?

Recall propositional language $\mathcal{L} = \langle \mathbf{C}, \{p_i \mid i \in \mathrm{N}\}, \emptyset, \mathbf{ar}\rangle$, $\mathbf{ar}(p_i) = 0$

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- If $\varphi \in T$, then $T \models_{\mathbb{K}} \varphi$ (Reflexivity)
- If $S \models_{\mathbb{K}} \psi$ for each $\psi \in T$ and $T \models_{\mathbb{K}} \varphi$, then $S \models_{\mathbb{K}} \varphi$ (Cut)
- If $T \models_{\mathbb{K}} \varphi$, then $\sigma[T] \models_{\mathbb{K}} \sigma(\varphi)$ for all substitutions σ (Structurality)

where substitution is any mapping from $\{p_i \mid i \in \mathrm{N}\}$ to Form

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But $\models_{\mathbb{K}}$ need not be finitary, i.e., we do not have

$$T \vdash \varphi \text{ implies } T' \vdash \varphi \text{ for some finite } T' \subseteq T$$

This is the case e.g. for $\mathbb{K} = \{[0,1]_{Ł}\}$.

We want propositional logics to be a bit 'better' (DC6)

Assume, from now on, that there is a binary operation $\to$ in $\mathcal{L}$ st: DC6

$$x \to y \geq \overline{1} \quad \text{iff} \quad x \leq y \qquad \text{for each } \boldsymbol{A}$$

y is 'better' than x IFF $x \to y$ is 'good'

Then $\models_{\mathbb{K}}$ is algebraically implicative à la C and Noguera
and, if finitary, algebraizable logic à la Blok and Pigozzi

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Then $\models_{\mathbb{K}}$ is algebraically implicative à la C and Noguera and, if finitary, algebraizable logic à la Blok and Pigozzi, i.e., we will always have:

$$\begin{array}{lll} \models_{\mathbb{K}} \varphi \to \varphi & \varphi, \varphi \to \psi \models_{\mathbb{K}} \psi & \varphi \to \psi, \psi \to \chi \models_{\mathbb{K}} \varphi \to \chi \\ \varphi \models_{\mathbb{K}} \overline{1} \to \varphi & \overline{1} \to \varphi \models_{\mathbb{K}} \varphi & \\ \models_{\mathbb{K}} \varphi \wedge \psi \to \varphi & \models_{\mathbb{K}} \varphi \wedge \psi \to \psi & \chi \to \varphi, \chi \to \psi \models_{\mathbb{K}} \chi \to \varphi \wedge \psi \\ \models_{\mathbb{K}} \varphi \to \varphi \vee \psi & \models_{\mathbb{K}} \psi \to \varphi \vee \psi & \varphi \to \chi, \psi \to \chi \models_{\mathbb{K}} \varphi \vee \psi \to \chi \end{array}$$

and for each n-ary $c \in \mathbf{C}$, formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i < n$:

$$\varphi \to \psi, \psi \to \varphi \models_{\mathbb{K}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$$

How to axiomatize $\models_{\mathbb{K}}$?

Lets us first restrict to propositional languages

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Lets us first restrict to propositional languages

Now I need to get a bit technical (sorry)

How to axiomatize $\models_{\mathbb{K}}$?

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A quasivariety is a class of algebras axiomatized by quasiidentities
formulas of the form $(\bigwedge_{i \leq n} \alpha_i \approx \beta_i) \to \varphi \approx \psi$

By $\mathbf{Q}(\mathbb{K})$ we denote the quasivariety generated by $\mathbb{K}$ i.e.,
the smallest class of algebras satisfying all quasiidentites valid in $\mathbb{K}$

Theorem (For propositional logic only)

$\models_{\mathbb{K}}$ *is finitary iff* $\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$.

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If $\models_{\mathbb{K}}$ is finitary, than it is axiomatized 'using' the quasiidentities
axiomatizing $\mathbf{Q}(\mathbb{K})$

1st axiomatizability result

Theorem (PC, C. Noguera. JSL 2015)

Let $\mathbb{K}$ be a quasivariety
and $\mathcal{AX}$ an arbitrary axiomatization of the propositional logic of $\mathbb{K}$.

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Then the following are equivalent:

- $T \models_{\mathbb{K}} \varphi$
- *there is a proof of φ from T in the axiomatic system:*

(P)	*first-order substitutions of axioms and rules of $\mathcal{AX}$*	
$(\forall 1)$	$\vdash (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	t *substitutable for* x *in* φ
$(\exists 1)$	$\vdash \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$	t *substitutable for* x *in* φ
$(\forall 2)$	$\chi \to \varphi \vdash \chi \to (\forall x)\varphi$	x *not free in* χ
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Why only for such big $\mathbb{K}$s?

Especially if we know that for finitary $\models_{\mathbb{K}}$ and propositional languages:

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Then $\varphi \vee \psi \models_{\mathbb{K}} ((\forall x)\varphi) \vee \psi$ but $\varphi \vee \psi \not\models_{\mathbf{Q}(\mathbb{K})} ((\forall x)\varphi) \vee \psi$

Other example: the set $\{\varphi \mid \models_{[0,1]_{Ł}} \varphi\}$ is coNP-complete for propositional languages but Π_2-complete in general while $\{\varphi \mid \models_{\mathbf{Q}([0,1]_{Ł})} \varphi\}$ is Σ_1-complete

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But at least we will have soundness:

$$\models_{\mathbb{K}} \quad \supseteq \quad \models_{\mathbf{Q}(\mathbb{K})} \quad = \quad \vdash$$

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A function $f\colon A \to B$ is a σ-embedding if:

- f is one-one
- $f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))$ for each n-ary $c \in \mathbf{C}$
- for each $X \subseteq A$, if $\inf_{\leq^A} X$ exists, then $f(\inf_{\leq^A} X) = \inf_{\leq^B} f[A]$.
- for each $X \subseteq A$, if $\sup_{\leq^A} X$ exists, then $f(\sup_{\leq^A} X) = \sup_{\leq^B} f[A]$.

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2nd axiomatizability result

Theorem (PC, C. Noguera. JSL 2015)

Let $\mathbb{K}$ be a class of all chains in the quasivariety $\mathbb{K}$ generates and $\mathcal{AX}$ an arbitrary axiomatization of the propositional logic of $\mathbb{K}$. Then the following are equivalent:

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$(\forall 2)^{\vee}$	$(\chi \to \varphi) \vee \psi \vdash (\chi \to (\forall x)\varphi) \vee \psi$	x not free in χ and ψ
$(\exists 2)^{\vee}$	$(\varphi \to \chi) \vee \psi \vdash ((\exists x)\varphi \to \chi) \vee \psi$	x not free in χ and ψ

When can we axiomatize a logic based on a ‘smaller’ class of chains? (let us restrict to countable predicate languages)

Theorem

Let $\mathbb{K}$ be a class of $\mathcal{L}$-chains and for each countable chain $\boldsymbol{A} \in \mathbf{Q}(\mathbb{K})$ there is a σ-embedding of $\boldsymbol{A}$ into some $\boldsymbol{B} \in \mathbb{K}$. Then

$$\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$$

Again, this condition is not necessary, only sufficient.

A summary of this section

We have designed 'logics of graded notions' based on design choices:

DC1 the syntax is almost classical; we only consider an arbitrary set $\mathcal{L}$ of propositional connectives

DC2 connectives have truth-functional interpretations

DC3′ some grades are better than others and for each two grades there is the best (worst) grade worse (better) than both of them $\wedge, \vee \in \mathcal{L}$

DC4 quantifiers are interpreted using infima and suprema over the set of instances of the formulas quantified

DC5 some grades are 'good'; the logic/consequence is the transition of 'goodness'; and there is the least 'good' grade $\overline{1} \in \mathcal{L}$

DC6 the order of grades and the set of good grades are mutually definable using implication $\to \in \mathcal{L}$

We have axiomatized, in some cases, the resulting logics

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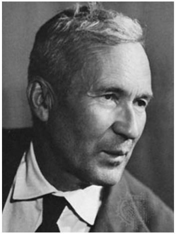
Outline

Three wise men



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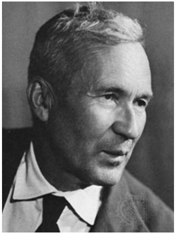

Andrey Kolmogorov

Jan Łukasiewicz

Petr Hájek

Let us start with semantics . . .

A *Kripke model* is a system $\mathbf{M} = \langle W, \langle e_w \rangle_{w \in W} \quad \rangle$ where

- W is a set (of possible worlds)
- for each $w \in W$, e_w is a classical evaluation

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A *probabilistic Kripke model* is a system $\mathbf{M} = \langle W, \langle e_w \rangle_{w \in W}, \mu \rangle$ where

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- μ is a finitely additive probability measure defined on a sublattice of 2^W which contains the set $\{w \mid e_w(\varphi) = 1\}$ for each classical φ

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Example: set $W = \{1, \ldots, 6\}$ and consider variables $v_1, \ldots, v_6, v_{\text{odd}}, v_{\geq 5}$

$$e_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{othrw.} \end{cases} \quad e_i(v_{\text{odd}}) = \begin{cases} 1 & \text{if } i \in \{1,3,5\} \\ 0 & \text{othrw.} \end{cases} \quad e_i(v_{\geq 5}) = \begin{cases} 1 & \text{if } i \geq 5 \\ 0 & \text{othrw.} \end{cases}$$

$$\mu(A) = \frac{|A|}{6}$$

Syntax: ‘notions’ could be quite different

We distinguish three different kinds of formulas:

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We distinguish three different kinds of formulas:

- classical: built from atoms using classical connectives: $\rightarrow, \neg, \vee$
- atomic fuzzy: of the form $\Box\varphi$, for a classical formula φ
- fuzzy: built from atomic ones using Łukasiewicz connectives $\rightarrow_{Ł}, \neg_{Ł}, \leftrightarrow_{Ł}, \oplus, \ominus$

Computing truth values in $\mathbf{M} = \langle W, \langle e_w \rangle_{w \in W}, \mu \rangle$

Truth value of non-modal formula φ in a world φ:

$$||\varphi||_{\mathbf{M},w} = e_w(\varphi)$$

Truth value atomic modal formula $\Box\varphi$ in $\mathbf{M}$:

$$||\Box\varphi||_{\mathbf{M}} = \mu(\{w \mid ||\varphi||_{\mathbf{M},w} = 1\})$$

Truth value other modal formulas in $\mathbf{M}$:

$$||\neg_{Ł}\gamma||_{\mathbf{M}} = 1 - ||\gamma||_{\mathbf{M}}$$
$$||\gamma \to_{Ł} \delta||_{\mathbf{M}} = \min\{1, 1 - ||\gamma||_{\mathbf{M}} + ||\delta||_{\mathbf{M}}\}$$
$$||\gamma \leftrightarrow_{Ł} \delta||_{\mathbf{M}} = 1 - \max\{||\gamma||_{\mathbf{M}} - ||\delta||_{\mathbf{M}}, ||\delta||_{\mathbf{M}} - ||\gamma||_{\mathbf{M}}\}$$
$$||\gamma \oplus \delta||_{\mathbf{M}} = \min\{1, ||\gamma||_{\mathbf{M}} + ||\delta||_{\mathbf{M}}\}$$
$$||\gamma \ominus \delta||_{\mathbf{M}} = \max\{0, ||\gamma||_{\mathbf{M}} - ||\delta||_{\mathbf{M}}\}$$

Axiomatization

Theorem

Let δ *be a fuzzy formula. Then* $||\delta||_{\mathbf{M}} = 1$ *for each probabilistic Kripke model* $\mathbf{M}$ *if and only if* δ *is provable in the axiomatic system:*

- *the axioms of classical logic for classical formulas*
- *the axioms of Łukasiewicz logic for fuzzy formulas*
- *modus ponens for both classical and Łukasiewicz implication*
- *additional rules:*

  from $\varphi \to \psi$ *infer* $\Box\varphi \to \Box\psi$ *from* φ *infer* $\Box\varphi$

- *additional axioms:*

  $\Box(\neg\varphi) \leftrightarrow_{Ł} \neg_{Ł}\Box\varphi$ $\Box(\varphi \vee \psi) \leftrightarrow_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))$

Outline

Future work: the ‘logical’ part

- extending the scope of our results
- generalizing other usual classical results
  - developing model-theory of our structures
  - studying the usual strengthenings of classical FO
- studying genuinely ‘non-classical’ aspects of our approach:
  - safe structures
  - unusual forms of Skolemization, Herbrand theorem etc.
  - witnessed structures
  - generalized quantifiers
- exploring connections to other approaches to non-classical FOL:
  - those close in spirit to ours; e.g. Ono’s treatment of first-order substructural logics
  - those based on some kind of Kripke semantics
  - categorial approaches
  - game-theoretic semantics
  - continuous model theory

Future work: the vision

$$\infty$$

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$$\infty$$

Join us, and together we can turn it into a program

A bit of propaganda: The Czech Society for Cybernetics and Informatics

The society objectives center on support and promotion of cybernetics, informatics and related fields, advancing the professional standing of its members, providing services to its members, and support of conferences, seminars and other activities.

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Working group for logic, probability and reasoning studies both

- the theoretical foundations of logic and probability theory and
- also their applications, mostly in
  - modeling of human reasoning and interaction and social behavior
  - computer science
  - artificial intelligence
  - data mining and
  - economy